Consistent canonical quantization of general relativity in the space of Vassiliev knot invariants

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We present a quantization of the Hamiltonian and diffeomorphism constraint of canonical quantum gravity in the spin network representation. The novelty consists in considering a space of wavefunctions based on the Vassiliev knot invariants. The constraints are finite, well defined, and reproduce at the level of quantum commutators the Poisson algebra of constraints of the classical theory. A similar construction can be carried out in $2+1$ dimensions leading to the correct quantum theory.

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The Ashtekar new variables [1] describe general relativity as a theory of a connection, having the same kinematical phase space as a Yang–Mills theory. The canonical conjugate pair is given by a set of (densitized) triads $\tilde{E}^a_i$ and an SU(2) connection A^i_a. This allowed to describe the theory in terms of holonomies [2] leading to the development of the loop representation, and later the spin network representation [3]. These representations encode in a natural way the diffeomorphism invariance of the theory through the notion of knot invariance. The dynamics of the theory, embodied in the Hamiltonian constraint, remained elusive. The quantization of this constraint led to the so-called Wheeler-DeWitt equation in the traditional formulation of general relativity. This is a non-polynomial equation and presents several challenges as a quantum field theory, since the usual techniques for regularizing operators introduce fiducial background metric structures that are incompatible with the general covariance of the theory. In terms of the Ashtekar new variables an important step forward was realized when Thiemann [4] showed how to write the Hamiltonian constraint as a scalar on the manifold. This raised hopes that a natural realization in terms of spin networks could be achieved. Thiemann represented the action of this constraint on diffeomorphism invariant states. He showed that the constraint commuted with itself, as one expects in a diffeomorphism invariant context. Moreover, Thiemann's formulation took place in the context of the real version of the Ashtekar variables introduced by Barbero [5], bypassing the controversial issue of the "reality conditions". The Hamiltonian considered corresponded to the usual real, Lorentzian general relativity.

In this paper we will present a realization of the Hamiltonian constraint in terms of a different space of wavefunctions, associated with the Vassiliev knot invariants. A distinctive feature of these wavefunctions is that they are "loop differentiable". The loop derivative [6] is the derivative that arises in the space of functions of loops when one considers the change in wavefunctions due to the addition of an infinitesimal loop. In the context of holonomies, this derivative encodes the information of the curvature tensor F_{ab}. There is a well known difficulty with computing this derivative in the context of knot invariants, since due to the diffeomorphism symmetry there is no notion of "infinitesimal" loop. Therefore one cannot compute the limit involved in the derivative in a direct way. In the case of Vassiliev invariants one can assign a value to this limit recalling the relationship between them and the expectation value of the Wilson loop in a Chern–Simons theory,

$$E(s,\kappa) = \int DA \exp\left(-\frac{1}{\kappa}\mathrm{Tr}(A\wedge\partial A + \frac{2}{3}A\wedge A\wedge A)\right) W_s[A] \tag{1}$$

where s is a spin network (a multivalent graph with holonomies in representations of SU(2) associated with each edge) and $W_s[A]$ is an SU(2) invariant obtained by interconnecting the holonomies along the edges with appropriate intertwiners constructed with invariant tensors in the group. It is a natural generalization to the spin network context of the "Wilson loop" (trace of the holonomy) one constructs with ordinary loops. The quantity $E(s,k)$ is an infinite series in powers of $\frac{1}{k}$, and is a (framing dependent) knot invariant. This invariant was first considered as connected with a Chern–Simons theory by Witten [7] in the context of loops and remarkably, also in the context of spin networks [8,9]. In the context of loops this invariant is associated with the evaluation for a particular value of the variable of the Kauffman polynomial. The coefficients in the infinite series are all knot invariants and one can isolate within these coefficients the elements of a basis of framing independent invariants called the Vassiliev invariants when restricted to ordinary loops. This construction can be extended to the spin network context, as we showed in two recent papers [10,11]. We will refer to the resulting invariants as Vassiliev invariants (including the framing dependent ones), although it should be noticed that this is a generalization of the usual notion of Vassiliev invariant, which is

customarily introduced for ordinary non-intersecting loops. One can evaluate the loop derivative on these invariants and one is left with a simple formula [10],

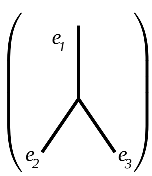
$$\Delta_{ab}(\pi_o^x)E(s,\kappa) = \kappa \sum_{e_k} (-1)^{2(J_j+J_k)} \Lambda_{J_j J_k} \epsilon_{abc} \int_{e_k} dy^c \delta^3(x-y) E\left(s',\kappa\right). \tag{2}$$

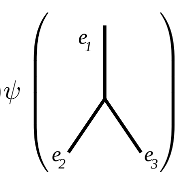
where s' is a new spin network obtained by interconnecting in a certain way the original spin network s with the path π on which the loop derivative Δ_{ab} depends, and $\Lambda_{J_j J_k}$ is a group factor dependent on the valences J_j and J_k of the lines e_j and e_k. The action of the derivative is distributional, as one would expect in a diffeomorphism invariant context. A similar action is obtained not just for the infinite series E but also for each individual coefficient and its framing dependent and framing independent portions.

In terms of the loop derivative we just discussed one can now obtain an action for the Hamiltonian constraint in the scalar version introduced by Thiemann [4]. We will only discuss for simplicity here the action on trivalent spin networks and we will concentrate on the "Euclidean" portion of the constraint. Thiemann has shown how if one has the action of this portion one can construct the rest of the full Lorentzian Hamiltonian constraint. Classically, the constraint is written as [4],

$$H(N) = \frac{2}{G} \int d^3x N(x) \{A_a^i, V\} F_{bc}^i \tilde{\epsilon}^{abc}, \tag{3}$$

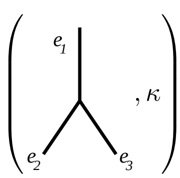
where V is the volume of the manifold and G is Newton's constant. At a quantum level, one introduces a triangulation adapted to the spin network of the state one is acting upon, replaces the Poisson bracket by a commutator, and represents the connection as an infinitesimal holonomy. In the context of trivalent intersections only one term in the commutator is non-vanishing, and one gets for the Hamiltonian [11],

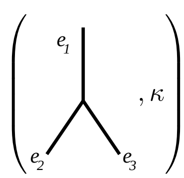
$H(N)\psi$ 

$$= \frac{8}{3G} \lim_{\epsilon\to 0} \int d^3y \sum_{v\in s} \epsilon_{ijk} \int_{e_i} du^a \int_{e_j} dw^b \chi(u,w,y;v) N(y) \rho(J_1,J_2,J_3) \Delta_{ab}^{(k)}(\pi_v^y)\psi$$


, (4)

where ρ is a group factor dependent on the valences of the three incoming lines at the intersection. The action of the Hamiltonian is only non-vanishing at intersections. The function χ is a regulator that restricts the integrals in u, w to the tetrahedra surrounding the vertex v and fixes the point y to the vertex v, a concrete realization is $\chi(y,z,w) = \Theta_\Delta(y,v)\Theta_\Delta(z,v)\Theta_\Delta(w,v)/\mathcal{V}\epsilon^3$ where the Theta functions are one if the first argument is within any of the eight tetrahedra surrounding the vertex v and zero otherwise, and the volume of each tetrahedra is given by $\epsilon^3\mathcal{V}$. This expression is quite similar to the original proposal for a (doubly densitized) Hamiltonian in the loop representation in terms of the loop derivative [12]. If one particularizes this expression to the expectation value of the Wilson net, one gets a very compact expression [11],

$H(N)E$ 

$$, \kappa \quad = -\frac{\kappa}{3G} \sum_{v\in s} N(v) \nu_{(J_1 J_2 J_3)} E$$


$, \kappa$, (5)

where $\nu_{J_i J_j J_k}$ is a group factor. From this expression one can derive the action of the Hamiltonian on a given Vassiliev invariant; it turns out to produce an invariant of one order less. It is quite remarkable that the action of the loop derivative in a space of diffeomorphism invariant functions yields a finite well defined expression for the constraint. For intersections of valences higher than three the action of the Hamiltonian ceases to be just a prefactor, but it still can be written explicitly. One can also introduce a diffeomorphism constraint,

$$C(\vec{N})\Psi(s) = \sum_k \lim_{\epsilon\to 0} \int d^3x \int_{e_k} dy^b \frac{(N^a(x) + N^a(y))}{2} f_\epsilon(x,y) \Delta_{ab}(\pi_y^x)\Psi(s). \tag{6}$$

where $f_\epsilon(x,y)$ is a regularization of the Dirac delta. Acting on Vassiliev invariants, one can explicitly check via a detailed calculation [11] that the constraint vanishes identically, as one would expect since the wavefunctions are diffeomorphism invariant [13].

As we see from equation (4), the action of the Hamiltonian constraint on a Vassiliev invariant produces a prefactor that depends on the location of the vertices times a group prefactor times a Vassiliev invariant. The location of the vertex is determined by the the intersection of the edges of the spin network. The latter are modified by the loop derivative, and as a consequence the loop derivative acts on functions of the position of the vertices. The loop derivative leaves the group factors unchanged. Therefore the action of the Hamiltonian produces as a result a function that is not diffeomorphism invariant but that is still loop differentiable, allowing one can compute the constraint algebra. We call these states generically $\psi(s, M, \Omega)$ where M is the function of the vertex and Ω the group factor. We can think of these states as the action of an operator $\hat{O}(M, \Omega)$ on $\psi(s)$. An explicit calculation [11] shows that,

$$C(\vec{N})O(M,\Omega)\psi(s) = O(N^a\partial_a M,\Omega)\psi(s) + O(M,\Omega)C(\vec{N})\psi(s). \tag{7}$$

That is, the diffeomorphism Lie-drags the prefactor and therefore acts geometrically. This ensures that the constraint algebra of diffeomorphisms is correctly implemented in this space. It also shows that the commutator of diffeomorphism and Hamiltonian is correct, that is, the Hamiltonian transforms covariantly.

To study the consistency of the commutator of two Hamiltonians with the classical Poisson relation $\{H(N), H(M)\} = C(q^{ab}V_a)$ where $V_a = M\partial_a N - N\partial_a M$, one needs to promote to a quantum operator the right-hand-side of the relation, which is proportional to the product of a diffeomorphism and the doubly-contravariant spatial metric. When one computes the right hand side, one finds that it vanishes identically on spin network states. This, in fact, can be tracked down to the vanishing of the double contravariant metric, which quantum mechanically can be written as [14,11],

$$\hat{q}^{ab}(z)\psi(s) = \lim_{\delta\to 0}\lim_{\epsilon\to 0}\frac{8}{9G^2}\sum_{v\in s}\int_{e_r} dy^a \int_{e_u} dw^b \epsilon^{pqr}\epsilon^{stu}\frac{\delta^6}{\epsilon^6}\Theta_\Delta(y,v)\Theta_\Delta(w,v)Q(e_p,e_q,e_s,e_t)\psi(s) \tag{8}$$

where the operator Q can be written in terms of the holonomies along edges incoming to the vertex and the volume operator and is finite for any spin network. If one assumes that the regularizations δ and ϵ are of the same order, the above expression is of order ϵ^2 (given by the two one dimensional integrals of Θ functions of size ϵ) and therefore vanishes. If one computes the doubly-covariant metric one finds that it diverges.

In spite of the fact that the loop derivative acts on the prefactor generated by the action of the Hamiltonian, when one computes the successive action of two Hamiltonians a cancellation takes place [11] and the left hand side of the commutator equation vanishes and therefore the algebra is consistent.

There is regularization ambiguity in these expressions. A clear example of this is in the double contravariant metric where there are two limits and one could choose to carefully "tune" them in order to end with a non-vanishing expression. The price to pay is that the non-vanishing expression depends on the background structures used in the regularization. This is not surprising. In the spin network representation we are in a manifold without a pre-determined metric. The only information we have are the locations of intersections and the orientations of the lines entering (not their tangent vectors). This is insufficient information to construct a symmetric tensor. Therefore the expression for the metric was bound to either be zero or background dependent. Similar considerations hold for the covariant metric. A posteriori, the result we find via a careful regularization is what one should have intuitively expected.

We therefore have a non-trivial, well defined quantization of canonical general relativity with the space of states given by the Vassiliev invariants. The expressions of the constraints are relatively simple, well defined and finite. Moreover, one can compute the constraint algebra and it is consistent with the classical Poisson algebra. Notice that the realization of the constraints is "off shell" in the sense that we do not need to work with diffeomorphism invariant states from the outset, and in fact this is sensible since the Hamiltonian constraint does not map within such a space of states. These points (the space of states chosen and the fact that we have an infinitesimal generator of diffeomorphisms) distinguish our construction from that of Thiemann which operated on diffeomorphism invariant states. It has in common the fact that the Hamiltonian commutes with itself.

Should one worry about a theory of quantum gravity where the metric appears to vanish? This will largely depend on how the semi-classical limit is set up for the theory. As we argued above, the double contravariant metric could not be anything else but vanishing in the context of the spin network quantum theory. More meaningful physical operators (like the length, the area and curvature invariants [15]) are non-vanishing and the volume operator would also be non-vanishing if one included intersections beyond the trivalent ones. A correct semi-classical limit could be built in terms of these and other operators which are in no sense degenerate. Can one find solutions to the Hamiltonian constraint? We can already construct several. If one considers the framing independent Vassiliev invariants, one

can check that they are annihilated by the Hamiltonian constraint (in the context of trivalent intersections) [11]. What is lacking if one compares with the construction of Thiemann is to have an inner product that would allow us to characterize these and other states as normalizable. Other, more non-trivial solutions (some of them with a cosmological constant) are likely to be present, as is hinted by the results involving Chern–Simons states in the loop representation ([16], see also [10] for some results in terms of spin networks).

Thiemann's approach has also been studied in $2+1$ dimensions [17], and appears to lead to a satisfactory quantization, provided one chooses in an ad-hoc way an inner product that rules out certain infinite dimensional set of solutions. In a forthcoming paper we will discuss the quantization of $2+1$ dimensional gravity using an approach that has elements in common with the one we pursue here, in particular the requirement of loop differentiability of the states. We will see that this requirement limits us (at least for low valence intersections) to the correct solution space in a natural way.

Having a family of consistent theories provides a context for calculations that are of a more "kinematical" nature, like the calculations of the entropy of black holes [18]. It also provides a basis for calculations of semi-classical behavior that are more dependent on the dynamics of the theory [19]. It is expected that the theory could be coupled to matter following the ideas of Thiemann [20]. Deciding if one of these consistent theories is a physically realistic quantum theory of gravity will have to wait until testable predictions that involve the dynamics in a more elaborate way are worked out.

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